Math 279 Lecture 23 Notes

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1 Norms and Schauder Estimates for Hölder-Zygmund Spaces

1.1 Equivalence of definitions of Hölder-Zygmund spaces

Recall that for our function spaces, we are using Hölder (or rather Hölder-Zygmund) spaces, and one of the simplest regularity estimates that are available for elliptic/parabolic PDE are the Schauder-type estimattes. For example, if K is smooth off of 0, with

1. $\operatorname{supp} K \subseteq B_1(0)$
2. $|\partial^k K(x)| \lesssim |x|^{\beta-d-|k|}$,

our Schauder estimates assert that

$$u \in \mathcal{C}^\alpha \implies K * u \in \mathcal{C}^{\alpha+\beta}.$$

Recall that for $\alpha < 0$, $\mathcal{C}^\alpha$ consists of $u \in \mathcal{D}'$ such that

$$[u]_{\alpha,K} = \sup_{x \in K} \sup_{\delta \in (0,1]} \sup_{\varphi \in \mathcal{D}_r} \frac{|u(\varphi_x^\delta)|}{\delta^\alpha} < \infty \qquad \text{for every compact } K,$$

where $\mathcal{D}_r$ is the set $\varphi \in \mathcal{D}$ with $\operatorname{supp} \varphi \subseteq B_1(0)$ and $\|\varphi\|_{C^r} \leq 1$. Here, r is the smallest integer such that $r + \alpha > 0$. As for $\alpha = n + \alpha_0$ with $n \in \mathbb{N}$ and $\alpha \in (0,1)$, $\mathcal{C}^\alpha$ consists of functions u such that $\partial^k u$ exists for $|k| \leq n$, and $\partial^k u \in \mathcal{C}^{\alpha_0}$, the Hölder continuous functions of exponent α_0.

We now define

$$[u]_{\alpha,K} = \sup_{x \in K} \sup_{\delta \in (0,1]} \sup_{\varphi \in \mathcal{D}^{(n)}} \frac{|u(\varphi_x^\delta)|}{\delta^\alpha} < \infty \qquad \text{for every compact } K,$$

where now $\mathcal{D}^{(n)}$ is the set of $\varphi \in \mathcal{D}$ such that $\operatorname{supp} \varphi \subseteq B_1(0)$, $\|\varphi\|_{L^\infty} \leq 1$, and $\int \varphi P = 0$ for every polynomial P of degree at most n. Let us write $\widehat{\mathcal{C}}^\alpha$ for the set of distributions u for which $[u]_{\alpha,K} < \infty$ for every compact K.

Proposition 1.1. *$\widehat{\mathcal{C}}^\alpha = \mathcal{C}^\alpha$, and they are isomorphic.*

Proof. $\mathcal{C}^\alpha \subseteq \widehat{\mathcal{C}}^\alpha$ by Taylor expansion. For the converse, assume that $u \in \widehat{\mathcal{C}}^\alpha$, and pick $\rho \in \mathcal{D}$ with $\int \rho = 1$. Recall that $u = \lim_{\varepsilon \to 0} u * \rho^\varepsilon$ and that $u * \rho^\varepsilon$ is a smooth function for each ε. Then using a telescoping sum, we can write

$$\begin{aligned} u - u(\rho_x^\delta) &= \sum_{n=0}^\infty (u(\rho_x^{2^{-n-1}\delta}) - u(\rho_x^{2^{-n}\delta})) \\ &= \sum_{n=0}^\infty (u(\rho_x^{2^{-n-1}\delta}) - u(\rho_x^{2^{-n}\delta})) \\ &= \sum_{n=0}^\infty u((\rho^{1/2} - \rho)_x^{2^{-n}\delta}). \end{aligned}$$

Let us first assume that $n = 0$, i.e. $\alpha \in (0, 10$, so that $\rho^{1/2} - \rho \in \mathcal{C}^{(0)}$. This would allow us to assert that

$$|u((\rho^{1/2} - \rho)_x^{2^{-n}\delta})| \lesssim (2^{-n}\delta)^\alpha.$$

This, in particular, implies that the above sum is uniformly convergent, so u must be a function. On the other hand, this estimate implies that

$$|u(x) - u(\rho_x^\delta)| \lesssim \delta^\alpha.$$

Finally,

$$\begin{aligned} |u(x) - u(y) &\lesssim \delta^\alpha + |u(\rho_x^\delta) - u(\rho_y^\delta)| \\ &= \delta^\alpha + |u(\rho_x^\delta - \rho_y^\delta)|. \end{aligned}$$

Observe that

$$(\rho - \rho_a)_x^\delta = \rho_x^\delta - \rho_{x+\delta a}^\delta.$$

Hence, if we choose $a = \frac{y-x}{\delta}$, then we get $\rho_x^\delta - \rho_y^\delta$. So we may select $\delta = |x - y|$ to assert

$$|u(\rho_x^\delta - \rho_y^\delta)| \lesssim \delta^\alpha = |x - y|^\alpha.$$

Thus,

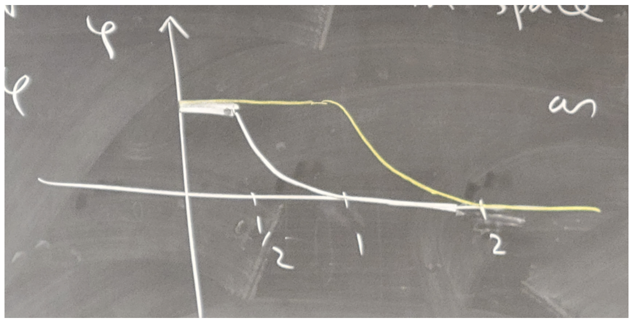
$$|u(x) - u(y)| \lesssim |x - y|^\alpha,$$

as desired.

This completes the proof when $n = 0$. For higher n, we integrate by parts. For example, if $n = 1$, then we can show that $\partial_{x_i} u \in \mathcal{C}^\alpha$ for each i by the previous case. $\square$

1.2 Schauder estimates for Hölder-Zygmund spaces

Now we focus on our Schauder estimate. We use a Paley-Littlewood type expansion of the kernel K but in the space variable. To prepare for this, start with a smooth function φ with $\varphi = 1$on $(0, 1/2]$ and $\operatorname{supp} \varphi \subseteq [0, 1]$. Then set $\psi(x) - \varphi(x/2) - \varphi(x)$, so that $\operatorname{supp} \psi \subseteq [1/2, 2]$.



Further define $\psi_n(x) = \psi(2^n x)$. Observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \psi_n(x) &= \sum_{n=0}^{\infty} (\varphi(2^{n-1}x) - \varphi(2^n x)) \\ &= \varphi(2^{-1}x),\end{aligned}$$

which equals 1 in $(0, 1]$. Observe that $\operatorname{supp} \psi_n \subseteq [2^{-n-1}, 2^{-n+1}]$, and if we write

$$K = \sum_{n=0}^{\infty} (\underbrace{K(x)\psi_n(|x|)}_{K_{n-1}}) = \sum_{n=-1}^{\infty} K_n(x)$$

with $K_n \subseteq \{x : |x| \in [2^{-n-2}, 2^{-n}\} \subseteq B_{2^{-n}}(0)$, then by our assumption on K,

$$|K_n(x)| \lesssim 2^{-n(\beta-d)}, \qquad |\partial^k K_n(x)| \lesssim 2^{-n(\beta-|k|-d)}.$$

More conveniently, we can think of this as

$$K = \sum_{n=-1}^{\infty} 2^{-n\beta}(2^{n\beta} K_n),$$

where the part in the parentheses looks like a Dirac delta. Here, the kernel K is a function with $K \in C^\infty(\mathbb{R}^d \setminus \{0\})$, $\operatorname{supp} \subseteq B_1(0)$, $|\partial_k K(x)| \lesssim |x|^{\beta-d-|k|}$.

We can now study $u \mapsto u * K$, where $u \in \mathcal{C}^\alpha$. We wish to show that $u * K \in \mathcal{C}^{\alpha+\beta}$.

$$(u * K)(\xi) = \text{“} \iint u(x-y)K(y)\xi(x)\, dx\, dy\text{”}$$

$$
\begin{aligned}
&= \text{“}\iint u(y)\widetilde{K}(y-x)\xi(x)\,dx\,dy\text{”} \\
&= u(\widetilde{K} * \xi) \\
&= \sum_{n=-1}^{\infty} 2^{-n\beta} u(2^{n\beta}\widetilde{K}_n * \xi).
\end{aligned}
$$

We wish to use $u \in \mathcal{C}^\alpha$ to get an estimate for $u(2^{n\beta}\widetilde{K}_n * \xi)$, where $\xi = \psi_a^\delta$ with ψ that satisfies certain conditions. For example, if $\alpha + \beta < 0$, then $\psi \in \mathcal{D}_r$; otherwise, we need some polynomial condition. We wish to use two pieces of information, $u \in \mathcal{C}^\alpha$ and the bounds on K_n. $\operatorname{supp} \xi \subseteq B_\delta(a)$, and $\operatorname{supp}(2^{n\beta}K_n) \subseteq B_{2^{-n}}(0)$, and we have the bounds $|\xi| \lesssim \delta^{-d}$ and $|2^{n\beta}K_n| \lesssim (2^{-n})^d$, which can be used to assert that $\operatorname{supp} \xi * K_n \subseteq B_{\delta+2^{-n}}(a)$ and

$$|u(2^{-\beta}K_n * \xi)| \lesssim (\delta + 2^{-n})^\alpha.$$

We are assuming $\alpha + \beta < 0$, so

$$
\begin{aligned}
|(u * K)(\xi)| &\lesssim \sum_n (\delta + 2^{-n})^\alpha 2^{-n\beta} \\
&= \sum_{n:2^{-n}<\delta} + \sum_{n:2^{-n}>\delta} \\
&\leq \delta^\alpha \sum_{2^{-n}<\delta} 2^{-n\beta} + \sum_{2^{-n}>\delta} 2^{-n(\alpha+\beta)} \\
&\lesssim \delta^\alpha \delta^\beta + \sum_{n \leq \log_2 \frac{1}{\delta}} (2^{-(\alpha+\beta)})^n \\
&\lesssim \delta^{\alpha+\beta} + (2^{-(\alpha+\beta)})^{\log_2 \frac{1}{\delta}} \\
&= 2\delta^{\alpha+\beta},
\end{aligned}
$$

as desired.

Next, let us examine the case of $\alpha + \beta > 0$. In this case, we assume that $\int \psi P\,dx = 0$, P is a polynomial, and $\deg P \leq \alpha + \beta$. We need to use the latter condition to improve our second bound (even when $\alpha < 0$). This can be achieved by subtracting a suitable polynomial P from $2^{n\beta}K_n$ because ξ is orthogonal to such polynomials. We omit the rest of the details.